Construction of analytic function $f(z) = u(x, y) + iv(x, y)$ by Milne-Thomson method

❖ If the real part $u(x, y)$ of f(z) is given.

Step-I: Find $\frac{\delta u}{\delta x}$ and $\frac{\delta u}{\delta y}$

Step-II: Find $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\delta u}{\delta x} - i\frac{\delta u}{\delta y}$ [by Cauchy-Riemann equations $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$]

Step-III: Put x=z and y=o in $f'(z)$

Step-IV: Integrate $f'(z)$ to obtain f(z)

❖ If the imaginary part $v(x, y)$ of f(z) is given.

Step-I: Find $\frac{\delta v}{\delta x}$ and $\frac{\delta v}{\delta y}$

Step-II: Find $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\delta v}{\delta y} + i\frac{\delta v}{\delta x}$ [by Cauchy-Riemann equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$]

Step-III: Put x=z and y=o in $f'(z)$

Step-IV: Integrate $f'(z)$ to obtain f(z)

❖ Find the analytic function f(z)= u + iv of which the imaginary part is
$\mathbf{v(x, y) = e^x(x\,siny + y\,cosy)}$

Solution:

$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}\{\mathbf{e^x(x\,siny + y\,cosy)}\} = \mathbf{e^x(x\,siny + y\,cosy) + e^x siny}$

$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}\{\mathbf{e^x(x\,siny + y\,cosy)}\} = \mathbf{e^x(x\,cosy + cosy - y\,siny)}$

Then $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

$= \frac{\delta v}{\delta y} + i\frac{\delta v}{\delta x}$ [by Cauchy-Riemann equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$]

$= \{\mathbf{e^x(x\,cosy + cosy - y\,siny)}\} + \mathbf{i\,e^x(x\,siny + y\,cosy) + e^x siny}$

Put x=z and y=o in $f'(z)$ we get,

$f'(z) = +\{\mathbf{e^z(z\,.\,1 + 1 - 0)}\} + \mathbf{i\{e^z(0 + 0) + 0\} = e^z(z + 1)}$

Integrate $f'(z)$ we get,

$f(z) = \int \mathbf{e^z(z + 1)dz}$

$= \mathbf{(z + 1)}\int \mathbf{e^z}\,dz - \int\left\{\frac{d}{dz}(z + 1)\ \int \mathbf{e^z}\,dz\right\}dz$

$= \mathbf{(z + 1)e^z} - \int \mathbf{e^z}\,dz$

$= \mathbf{e^z(z + 1) - e^z + c}$

$= \mathbf{ze^z + c}$

❖ Find the analytic function f(z)= u + iv of which the real part is $\mathbf{u(x, y) = e^x(x\,cos2y - y\,sin2y)}$

Solution:

$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\{\mathbf{e^x(x\,cos2y - y\,sin2y)}\} = \mathbf{e^x(x\,cos2y - y\,sin2y) - 2e^x cos2y}$

$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}\{\mathbf{e^x(x\,cos2y - y\,sin2y)}\} = \mathbf{e^x(-2x\,sin2y - 2y\,cos2y - y\,sin2y)}$

Then $f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

$= \frac{\delta u}{\delta x} - i\frac{\delta u}{\delta y}$ [By **Cauchy-Riemann** equations $\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$]

$= \mathbf{e^x(x\,cos2y - y\,sin2y) - 2e^x cos2y - ie^x(-2x\,sin2y - 2y\,cos2y - y\,sin2y)}$

Put x=z and y=o in $f'(z)$ we get,

$f'(z) = \{\mathbf{e^z(z\,.\,1 + 0) - 2\,e^z.1}\} - \mathbf{i}\{\mathbf{e^z(0 - 0 - 0)}\} = \mathbf{e^z(z - 2)}$

Integrate $f'(z)$ we get,

$f(z) = \int \mathbf{e^z(z - 2)dz}$

$= \mathbf{(z - 2)}\int \mathbf{e^z}\,dz - \int\left\{\frac{d}{dz}(z - 2)\ \int \mathbf{e^z}\,dz\right\}dz$

$= \mathbf{(z - 2)e^z} - \int \mathbf{e^z}\,dz$

$= \mathbf{e^z(z - 2) - e^z + c}$

$= \mathbf{(z - 3)e^z + c}$

❖ **Self exercise**

(i) Find the analytic function f(z)= u + iv of which the real part is $u(x, y) = \mathbf{log\sqrt{x^2 + y^2}}$

(ii) Find the analytic function f(z)= u + iv of which the real part is $u(x, y) = \mathbf{x^2 + y^2}$

(iii) Find the analytic function f(z)= u + iv of which the imaginary part is $\mathbf{v(x, y) = -2sinx\,(e^y - e^{-y})}$